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Fourier expansion and integral representation generalized Apostol-type Frobenius–Euler polynomials

Alejandro Urieles[1*], William Ramírez[2], María José Ortega[2] and Daniel Bedoya[3]

*Correspondence:
alejandrourieles@mail.uniatlantico.edu.co
[1]Programa de Matemáticas, Universidad del Atlántico, Km 7 Vía Pto, Barranquilla, Colombia
Full list of author information is available at the end of the article

Abstract

The main purpose of this paper is to investigate the Fourier series representation of the generalized Apostol-type Frobenius–Euler polynomials, and using the above-mentioned series we find its integral representation. At the same time applying the Fourier series representation of the Apostol Frobenius–Genocchi and Apostol Genocchi polynomials, we obtain its integral representation. Furthermore, using the Hurwitz–Lerch zeta function we introduce the formula in rational arguments of the generalized Apostol-type Frobenius–Euler polynomials in terms of the Hurwitz zeta function. Finally, we show the representation of rational arguments of the Apostol Frobenius Euler polynomials and the Apostol Frobenius–Genocchi polynomials.

Keywords: Generalized Apostol Frobenius–Euler polynomials; Hurwitz zeta function; Fourier expansion; Generalized Apostol Frobennius–Euler numbers

1 Introduction

The Fourier series of a periodic function can be written exponentially as (see [9, p. 19, Eq. (2.2)])

$$f(x)=\sum_{n=-\infty}^{\infty}a_n e^{inwx};\qquad \left(w=\frac{2\pi}{T}\right),$$

the coefficients a_n and $\overline{a}_n$ are computed by

$$a_n=\frac{1}{T}\int_0^{\frac{2\pi}{w}}e^{-inwt}f(t)\,dt\quad\text{and}\quad \overline{a}_n=\frac{1}{T}\int_0^{\frac{2\pi}{w}}e^{inwt}f(t)\,dt.$$

Here $\overline{a}_n$ is the complex conjugate of a_n.

The Frobenius–Euler polynomials and the Frobenius–Euler numbers play an important role in the number of theories and classical analysis. In particular, the Frobenius–Euler polynomials appear in the integral representation of differentiable periodic functions since they are employed for approximating such functions in terms of polynomials (see [1, 4–6, 10, 13, 19]). The Frobenius–Euler polynomials $H_n(x;u)$ in the variable x are defined by means of the generating function (see [11, p. 268])

$$\left(\frac{1-u}{e^z-u}\right)e^{xz}=\sum_{n=0}^{\infty}H_n(x;u)\frac{z^n}{n!},\quad |z|<\left|\log(u)\right|, \tag{1}$$

when $x=0$, $H_n(u)$ denotes the so-called Frobenius–Euler numbers. $H_n(x;-1)=E_n(x)$ denotes the Euler polynomials (see [16, 17]).

The Fourier series representation of the Frobenius–Euler polynomials are given by (see [2, p. 8, Corollary 4])

$$H_n(x;u)=\frac{(u-1)}{u}u^x n!\sum_{k\in\mathbb{Z}}\frac{e^{2\pi ikx}}{[2\pi ik+\log(u)]^{n+1}}, \tag{2}$$

if $u,\in\mathbb{C}$ with $u\neq 1$, and $0<x<1$.

The Frobenius–Genocchi polynomials $\mathfrak{G}_n^F(x;u)$ in the variable x are defined by the generating function (see [2, p. 3, Definition 3])

$$\left(\frac{1-u}{e^z-u}\right)ze^{xz}=\sum_{n=0}^{\infty}\mathfrak{G}_n^F(x;u)\frac{z^n}{n!};\quad |z|<\left|\log(u)\right|, \tag{3}$$

when $x=0$, $\mathfrak{G}_n^F(u)$ denotes the so-called Frobenius–Genocchi numbers, then the Fourier series representation of (3) is given by

$$\mathfrak{G}_n^F(x;u)=\frac{(u-1)}{u}u^x n!\sum_{k\in\mathbb{Z}}\frac{e^{2\pi ikx}}{[2\pi ik+\log(u)]^{n}}. \tag{4}$$

Some authors have proved a Fourier series and integral representations for the Apostol–Euler polynomials and Apostol–Bernoulli polynomials by using the Lipschitz summation formula (see [14]). On the other hand, in [3] using the Cauchy residue theorem in the complex plane, the author proved a Fourier series for the Apostol–Bernoulli, Apostol–Genocchi and Apostol–Euler polynomials. Other authors revealed a Fourier expansion for Apostol Frobenius–Euler polynomials and Apostol Frobenius–Genocchi polynomials (cf. [2]). We recently studied the Fourier expansions for higher-order Apostol–Genocchi, Apostol–Bernoulli and Apostol–Euler polynomials (see [8]).

In this paper, we obtained the Fourier expansion of generalized Apostol-type Frobenius–Euler polynomials and its integral representation to show the explicit formula at rational arguments for these polynomials in terms of the Hurwitz zeta function. Also, we will show the integral representation of Apostol Frobenius–Euler, Apostol Frobenius–Genocchi, Frobenius–Genocchi, Frobenius–Euler polynomials, and give a new representation for the polynomials of Apostol–Euler and Apostol–Genocchi and some formulas in rational arguments.

This article is organized as follows. Section 2 contains the basic background about polynomials of Apostol-type Frobenius–Euler, Apostol Frobenius–Genocchi, Frobenius–Genocchi, Frobenius–Euler and generalized Apostol-type Frobenius–Euler polynomials in the variable x, parameters $\lambda,u\in\mathbb{C}$, $a,b,c\in\mathbb{R}^+$. In Sect. 3 are revealed the Fourier expansions for the generalized Apostol-type Frobenius–Euler polynomials, and several corollaries for other families of known polynomials. In Sect. 4, we obtain the integral representation of generalized Apostol-type Frobenius–Euler polynomials, that is, Theorem 4.1.

At the same time, we achieved the integral representation of the Frobenius–Euler and Frobenius–Genocchi polynomials, that is, Theorems 4.2 and 4.3. Also, Sect. 5 secures the explicit formula at rational arguments in terms of Hurwitz zeta function of generalized Apostol-type Frobenius–Euler polynomials, that is, Theorem 5.1. Finally, we obtained the formula in rational arguments for the Frobenius–Euler and Frobenius–Genocchi polynomials, that is, Theorems 5.2 and 5.3, respectively.

2 Background and previous results

Throughout this paper, we use the following standard notions: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers. For the complex logarithm, we consider the principal branch.

On the other hand, we have the well-known integral formula (see [15, p. 2198, Eq. (3.2)])

$$\int_0^\infty t^n e^{-at}\,dt = \frac{n!}{a^{n+1}}, \quad n \in \mathbb{N}_0; \Re e(a) > 0. \tag{5}$$

It is well known that Apostol-type Frobenius–Euler polynomials $H_n(x;u;\lambda)$ in the variable x are defined by means of the generating function (see [4, p. 164, Eq. (1.1)])

$$\left(\frac{1-u}{\lambda e^z - u}\right)e^{xz} = \sum_{n=0}^{\infty} H_n(x;u;\lambda)\frac{z^n}{n!}, \quad |z| < \left|\log\left(\frac{\lambda}{u}\right)\right|. \tag{6}$$

The Fourier series representation of the Apostol-type Frobenius–Euler polynomials is given by (see [2, p. 5, Theorem 1])

$$H_n(x;u;\lambda) = \frac{(u-1)}{u}\frac{u^x}{\lambda^x}n!\sum_{k\in\mathbb{Z}}\frac{e^{2\pi ikx}}{[2\pi ik - \log(\frac{\lambda}{u})]^{n+1}}. \tag{7}$$

Also, the Fourier series representation of Apostol-type Frobenius–Genocchi polynomials is given by (see [2, p. 13, Theorem 11])

$$\mathfrak{G}_n^F(x;u;\lambda) = \frac{(u-1)}{u}\frac{u^x}{\lambda^x}n!\sum_{k\in\mathbb{Z}}\frac{e^{2\pi ikx}}{[2\pi ik - \log(\frac{\lambda}{u})]^{n}}, \tag{8}$$

which makes sense if $u, \lambda \in \mathbb{C}$ with $u \neq 1$, $\lambda \neq 1$, $u \neq \lambda$ and $0 < x < 1$.

For parameters λ, $u \in \mathbb{C}$, $u \neq \lambda$ and $a, b, c \in \mathbb{R}^+$ whit $a \neq b$, of generalized Apostol-type Frobenius–Euler polynomials are defined by means of the following generating functions (see [18, p. 9, Definition 4.1]):

$$\left(\frac{a^z - u}{\lambda b^z - u}\right)c^{xz} = \sum_{n=0}^{\infty}\mathfrak{H}_n(x;a,b,c;u;\lambda)\frac{z^n}{n!}, \quad |z| < \left|\frac{\log(\frac{\lambda}{u})}{\ln b}\right|, \tag{9}$$

if $x = 0$ in (9) then we get $\mathfrak{H}_n(a,b,c;u;\lambda)$, which denotes the generalized Apostol-type Frobenius–Euler numbers (see [18, p. 9]).

For $n = 0$ and $a, b \in \mathbb{R}^+$, whit $a \neq b$, $u, \lambda \in \mathbb{C}$ with $u \neq \lambda$ it is then true that $\mathfrak{H}_n(a,b;u;\lambda) = \frac{1-u}{\lambda-u}$. For $n > 0$ we have (see [18, p. 10, Theorem 4.2])

$$\lambda\big(\ln b + \mathfrak{H}(a,b;u;\lambda)\big)^n - u\mathfrak{H}_n(a,b;u;\lambda) = (\ln a)^n.$$

As an example, the generalized Apostol-type Frobenius–Euler numbers and polynomials are (with the help of MAPLE) as follows.

The generalized Apostol-type Frobenius–Euler numbers:

$$\begin{aligned}
\mathfrak{H}_0(a,b;u;\lambda) &= \frac{1-u}{\lambda-u},\\
\mathfrak{H}_1(a,b;u;\lambda) &= \frac{\ln a-\lambda\ln b}{(\lambda-u)},\\
\mathfrak{H}_2(a,b;u;\lambda) &= \frac{(\ln a)^2-\lambda(\ln b)^2}{(\lambda-u)}-\lambda\ln b\frac{\ln a-\lambda\ln b}{(\lambda-u)^2}.
\end{aligned}$$

The generalized Apostol-type Frobenius–Euler polynomials:

$$\begin{aligned}
&\mathfrak{H}_0(x;a,b,c;u;\lambda) = \frac{1-u}{\lambda-u},\\
&\mathfrak{H}_1(x;a,b,c;u;\lambda) = x\frac{\ln c(1-u)}{\lambda-u}+\frac{\ln a-\lambda\ln b}{(\lambda-u)},\\
&\mathfrak{H}_2(x;a,b,c;u;\lambda)\\
&\quad = x^2\frac{(\ln c)^2(1-u)}{\lambda-u}-2x\ln c\frac{\ln a-\lambda\ln b}{(\lambda-u)}+\frac{(\ln a)^2-\lambda(\ln b)^2}{(\lambda-u)}-2\lambda\ln b\frac{\ln a-\lambda\ln b}{(\lambda-u)^2}.
\end{aligned}$$

These polynomials are commonly said to be of Euler type, and they have been studied by various authors in different applications of practical importance (see [1, 12, 21]).

On the other hand, the Hurwitz–Lerch zeta function $\Phi(z,s,a)$ is defined as (see [15, p. 296, Eq. (4.1)])

$$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s},\quad a\in\mathbb{C}\setminus\mathbb{Z}_0^-; s\in\mathbb{C}\text{ when }|z|<1 \tag{10}$$

and $\Re e(s)>1$ for every $|z|=1$.

For $z=1$ in (10) we have the Hurwitz zeta functions

$$\zeta(s,a)=\Phi(1,s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}. \tag{11}$$

Recently, there was defined a new family of Lerch-type zeta function, interpolating a certain class of higher-order Apostol-type numbers and Apostol-type polynomials (cf. [20]). We will use (10) and (11) in Theorems 5.1 and 5.2.

3 Fourier expansion of generalized Apostol-type Frobenius–Euler polynomials $\mathfrak{H}_n(x,a,b,c;u;\lambda)$

In this section, we get the Fourier expansions for the generalized Apostol-type Frobenius–Euler polynomials.

Theorem 3.1 *Let* $u,\lambda\in\mathbb{C}\backslash\{0;1\}u\neq\lambda$ *and* $a,b,c\in\mathbb{R}^+, 1\leq a\leq 1.1, b>1, 1<c\leq e, 0<x<0.9$, *we have*

$$\mathfrak{H}_n(x,a,b,c;u;\lambda)=n!(\ln b)^n\left(\frac{u}{\lambda}\right)^{x\frac{\ln c}{\ln b}}\left[\frac{u-(\frac{u}{\lambda})^{\frac{\ln a}{\ln b}}}{u}\right]\sum_{k\in\mathbb{Z}}\frac{e^{\frac{2\pi kxi\ln c}{\ln b}}}{[2\pi ki-\log(\frac{\lambda}{u})]^{n+1}}. \tag{12}$$

Proof First we consider $f_n(z) = \frac{1}{z^{n+1}} \frac{a^z - u}{\lambda b^z - u} c^{xz}$ and the following integral:

$$\int_C f_n(z)\,dz, \tag{13}$$

over the circle $C = \{z : |z| \leq (2N + \varepsilon)\pi$ and $\varepsilon \in \mathbb{R}, (\varepsilon\pi i \pm \log(\frac{\lambda}{u}) \neq 0 \pmod{2\pi i})\}$.

The poles of the function $f_n(z)$ are given by

$$z_k = \frac{2\pi ki - \log(\frac{\lambda}{u})}{\ln b}, \quad k \in \mathbb{Z}.$$

With $z = 0$ a pole of order $n + 1$. From the Cauchy residue theorem we have (see [7, p. 112, Theorem 2.2])

$$\int_C f_n(z)\,dz = 2\pi i \left\{ \operatorname{Res}\bigl(f_n(z), z = 0\bigr) + \sum_{k\in\mathbb{Z}} \operatorname{Res}\bigl(f_n(z), z = z_k\bigr) \right\}. \tag{14}$$

We calculate $\operatorname{Res}(f_n(z), z = 0)$ and $\operatorname{Res}(f_n(z), z = z_k)$ as follows (see [7, p. 113, Proposition 2.4]):

$$\begin{aligned}
\operatorname{Res}\bigl(f_n(z), z = 0\bigr) &= \lim_{z\longrightarrow 0} \frac{1}{n!} \frac{d^n}{dz^n} \left[(z-0)^{n+1} \frac{1}{z^{n+1}} \sum_{m=0}^{\infty} \mathfrak{H}_{\mathrm{m}}(x, a, b, c; u; \lambda) \frac{z^m}{m!} \right] \\
&= \lim_{z\longrightarrow 0} \frac{1}{n!} \sum_{m=n}^{\infty} \mathfrak{H}_{\mathrm{m}}(x, a, b, c; u; \lambda) \frac{z^{m-n}}{(m-n)!} \\
&= \frac{\mathfrak{H}_{\mathrm{n}}(x, a, b, c; u; \lambda)}{n!}.
\end{aligned}$$

Also

$$\begin{aligned}
\operatorname{Res}\bigl(f_n(z), z = z_k\bigr) &= \lim_{z\longrightarrow z_k} (z - z_k)(z)^{-(n+1)} \frac{a^z - u}{\lambda b^z - u} c^{xz} \\
&= \frac{1}{z_k^{n+1}} \bigl(a^{z_k} - u\bigr) c^{xz_k} \lim_{z\longrightarrow z_k} \frac{z - z_k}{\lambda b^z - u} \\
&= \frac{1}{[\frac{2\pi kxi - \log(\frac{\lambda}{u})}{\ln b}]^{n+1}} \bigl(a^{\frac{2\pi kxi - \log(\frac{\lambda}{u})}{\ln b}} - u\bigr) c^{\frac{2\pi kxi + \log(\frac{u}{\lambda})^x}{\ln b}} \frac{1}{\lambda b^{\frac{2\pi kxi - \log(\frac{\lambda}{u})}{\ln b}} \ln b}.
\end{aligned}$$

So, in (14) we have

$$\begin{aligned}
\int_C f_n(z)\,dz = 2\pi i \Biggl\{ & \frac{\mathfrak{H}_{\mathrm{n}}(x, a, b, c; u; \lambda)}{n!} \\
& + \sum_{k\in\mathbb{Z}} \frac{(a^{\frac{2\pi kxi - \log(\frac{\lambda}{u})}{\ln b}} - u) c^{\frac{2\pi kxi + \log(\frac{u}{\lambda})^x}{\ln b}}}{[\frac{2\pi kxi - \log(\frac{\lambda}{u})}{\ln b}]^{n+1}} \frac{1}{\lambda b^{\frac{2\pi kxi - \log(\frac{\lambda}{u})}{\ln b}} \ln b} \Biggr\}.
\end{aligned}$$

Taking $N \to \infty$ it becomes $\int_C f_n(z)\,dz = 0$. So we have

$$\mathfrak{H}_{\mathrm{n}}(x, a, b, c; u; \lambda) = -n! c^{\log(\frac{u}{\lambda})^x} \sum_{k\in\mathbb{Z}} \frac{[a^{\frac{2\pi ki - \log(\frac{\lambda}{u})}{\ln b}} - u] c^{\frac{2\pi kxi}{\ln b}}}{[\frac{2\pi ki - \log(\frac{\lambda}{u})}{\ln b}]^{n+1} \lambda b^{\frac{2\pi ki - \log(\frac{\lambda}{u})}{\ln b}} \ln b}. \tag{15}$$

In (15), as a, b, c, are expressed in terms of exponential, we complete the proof. □

Corollary 3.1 *Let* $u, \lambda \in \mathbb{C}$ with $u \neq 1, \lambda \neq 1, u \neq \lambda; 0 < x < 1$ *and* $a = 1, b = c = e$, *we have*

$$\mathfrak{H}_n(x,1,e,e;u;\lambda) = \mathfrak{H}_n(x;u;\lambda) = \frac{(u-1)}{u}\frac{u^x}{\lambda^x} n! \sum_{k\in\mathbb{Z}} \frac{e^{2\pi ikx}}{[2\pi ik - \log(\frac{\lambda}{u})]^{n+1}}. \tag{16}$$

This is the Fourier expansion for Apostol-type Frobenius–Euler polynomials (*see* [2, *p*. 5, *Theorem* 1]).

Corollary 3.2 *Let* $u, \lambda \in \mathbb{C}$ with $u \neq 1, \lambda = 1, u \neq \lambda; 0 < x < 1$ *and* $a = 1, b = c = e$, *we have*

$$\mathfrak{H}_n(x,1,e,e;u;1) = \mathfrak{H}_n(x,1,u) = \mathfrak{H}_n(x,u) = \frac{(u-1)}{u} u^x n! \sum_{k\in\mathbb{Z}} \frac{e^{2\pi ikx}}{[2\pi ik + \log(u)]^{n+1}}. \tag{17}$$

This is the Fourier expansion for Frobenius–Euler polynomials obtained in (*see* [2, *p*. 8, *Corollary* 4]).

Corollary 3.3 *Let* $u, \lambda \in \mathbb{C}$ with $u \neq 1, u \neq \lambda; 0 < x < 1; a = 1, b = c = e; u = -1$, *we have*

$$\mathfrak{H}_n(x,1,e,e;-1;\lambda) = H_n(x,\lambda,-1) = \mathfrak{E}_n(x;\lambda) = 2n! \sum_{k\in\mathbb{Z}} \frac{e^{(2k-1)\pi ikx}}{[(2k-1)\pi i - \log(\lambda)]^{n+1}}. \tag{18}$$

This is the Fourier expansion for Apostol–Euler polynomials (*see* [14, *p*. 2196, *Eq*. (2.8)]).

4 Integral representation of the generalized Apostol-type Frobenius–Euler polynomials

In this section, we will show the integral representation of generalized Apostol-type Frobenius–Euler polynomials.

Theorem 4.1 *For* $n \in \mathbb{N}$ *and* $0 < x \leq 0,9$, $|\xi| < \frac{1}{2}$, $\xi \in \mathbb{R}$, $a, b, c \in \mathbb{R}^+$, $1 \leq a \leq 1, 1b > 1$ *and* $1 < c \leq e$,

$$\begin{aligned}\mathfrak{H}_n\big(x;a,b,c;u;-ue^{2\pi i\xi}\big) &= \Theta\left[\int_0^\infty \frac{D(n;x,v)(e^{2\pi(v-ix\frac{\ln c}{\ln b})}e^{2\xi\pi v} + e^{-2\xi\pi v})}{\cosh 2\pi v - \cos 2\pi x\frac{\ln c}{\ln b}} v^n\,dv\right] \\ &\quad + \Theta\left[\int_0^\infty \frac{iB(n;x,v)(e^{2\pi(v-ix\frac{\ln c}{\ln b})}e^{2\xi\pi v} - e^{-2\xi\pi v})}{\cosh 2\pi v - \cos 2\pi x\frac{\ln c}{\ln b}} v^n\,dv\right],\end{aligned} \tag{19}$$

where

$$\begin{aligned}\Theta &= \frac{1}{2}(\ln b)^n\left[\frac{u - e^{-2\pi ix\xi\frac{\ln a}{\ln b}}(-1)^{\frac{\ln a}{\ln b}}}{u}\right]e^{-(2\xi\pi ix\frac{\ln c}{\ln b})}, \\ D(n;x,v) &= \left[e^{\pi v}\cos\left(\pi x\frac{\ln c}{\ln b} - \frac{(n+1)\pi}{2}\right) + e^{-\pi v}\cos\left(\pi x\frac{\ln c}{\ln b} + \frac{(n+1)\pi}{2}\right)\right], \\ B(n;x,v) &= \left[e^{\pi v}\sin\left(\pi x\frac{\ln c}{\ln b} - \frac{(n+1)\pi}{2}\right) - e^{-\pi v}\sin\left(\pi x\frac{\ln c}{\ln b} + \frac{(n+1)\pi}{2}\right)\right].\end{aligned}$$

Proof From (12) and taking $\lambda = -ue^{2\pi i\xi}$, $k \longmapsto -k$ we have

$$\mathfrak{H}_n\big(x,a,b,c;u;-ue^{2\pi i\xi}\big)$$

$$= (\ln b)^n \left[\frac{u - e^{-2\pi i x\xi \frac{\ln a}{\ln b}}(-1)^{\frac{\ln a}{\ln b}}}{u}\right] e^{-\pi i x \frac{\ln c}{\ln b}} e^{-2\pi i \xi x \frac{\ln c}{\ln b}} \frac{n!}{(-\pi i)^{n+1}} \times \sum_{k\in\mathbb{Z}} \frac{e^{-2\pi i k x \frac{\ln c}{\ln b}}}{[2k + 2\xi + 1]^{n+1}}, \tag{20}$$

using (5) and

$$\left(\frac{-1}{i}\right)^{n+1} = e^{\frac{(n+1)\pi i}{2}}; \qquad (-1)^{(n+1)} = e^{-(n+1)\pi i},$$

we have

$$\begin{aligned}
&\mathfrak{H}_{\mathrm{n}}\left(x,a,b,c;u;-ue^{2\pi i\xi}\right) \\
&= \left[\frac{u - e^{-2\pi i x\xi \frac{\ln a}{\ln b}}(-1)^{\frac{\ln a}{\ln b}}}{u}\right] \frac{(\ln b)^n}{(-\pi i)^{n+1}} \sum_{k=0}^{\infty} e^{-(2k\frac{\ln c}{\ln b} + 2\xi\frac{\ln c}{\ln b} + \frac{\ln c}{\ln b})\pi i x} \int_0^\infty t^n e^{-(2k+2\xi+1)t}\,dt \\
&\quad + \left[\frac{u - e^{-2\pi i x\xi \frac{\ln a}{\ln b}}(-1)^{\frac{\ln a}{\ln b}}}{u}\right] \frac{(\ln b)^n}{(-\pi i)^{n+1}} (-1)^{n+1} \\
&\quad \times \sum_{k=0}^{\infty} e^{(2k\frac{\ln c}{\ln b} - 2\xi\frac{\ln c}{\ln b} - \frac{\ln c}{\ln b})\pi i x} \int_0^\infty t^n e^{-(2k-2\xi-1)t}\,dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathfrak{H}_{\mathrm{n}}\left(x,a,b,c;u;-ue^{2\pi i\xi}\right) \\
&= \left[\frac{u - e^{-2\pi i x\xi \frac{\ln a}{\ln b}}(-1)^{\frac{\ln a}{\ln b}}}{u}\right] \frac{(\ln b)^n e^{-2\xi\pi i x\frac{\ln c}{\ln b}}}{(-\pi i)^{n+1}} \int_0^\infty \frac{e^{-\pi i x\frac{\ln c}{\ln b}}}{e^{2t} - e^{-2\pi i x\frac{\ln c}{\ln b}}} e^{(1-2\xi)t} t^n\,dt \\
&\quad + \left[\frac{u - e^{-2\pi i x\xi \frac{\ln a}{\ln b}}(-1)^{\frac{\ln a}{\ln b}}}{u}\right] \frac{(\ln b)^n e^{-2\xi\pi i x\frac{\ln c}{\ln b}}}{(-\pi i)^{n+1}} (-1)^{n+1} \int_0^\infty \frac{e^{-\pi i x\frac{\ln c}{\ln b}}}{e^{2t} - e^{2\pi i x\frac{\ln c}{\ln b}}} e^{(3+2\xi)t} t^n\,dt,
\end{aligned}$$

then

$$\begin{aligned}
&\mathfrak{H}_{\mathrm{n}}\left(x;a,b,c;u;-ue^{2\pi i\xi}\right) \\
&= \frac{\Theta}{(\pi)^{n+1}} \left\{ \int_0^\infty e^{\frac{(n+1)\pi i}{2}} \mho(t;x,;b,c) e^{\pi i x\frac{\ln c}{\ln b}} e^{-(2\xi-1)t} t^n\,dt \right. \\
&\quad \left. + \int_0^\infty e^{\frac{-(n+1)\pi i}{2}} \frac{(e^{2\pi i x\frac{\ln c}{\ln b}} - e^{-2t})}{\cosh(2t) - \cos(2\pi x\frac{\ln c}{\ln b})} e^{-3\pi i x\frac{\ln c}{\ln b}} e^{(2\xi+3)t} t^n\,dt \right\},
\end{aligned}$$

where

$$\mho(t;x,;b,c) = \frac{(e^{-2\pi i x\frac{\ln c}{\ln b}} - e^{-2t})}{\cosh(2t) - \cos(2\pi x\frac{\ln c}{\ln b})}$$

and

$$\Theta = \frac{1}{2}(\ln b)^n \left[\frac{u - e^{-2\pi i x\xi \frac{\ln a}{\ln b}}(-1)^{\frac{\ln a}{\ln b}}}{u}\right] e^{-(2\xi\pi i x\frac{\ln c}{\ln b})}.$$

With $i^{n+1} = e^{\frac{(n+1)\pi i}{2}}$, $(-1)^{n+1} = e^{-(n+1)\pi i}$, and making the substitution $t = \pi v$ and simplifying, we complete the result. □

Corollary 4.1 *For $n \in \mathbb{N}$ and $0 \le x \le 1$, $|\xi| < \frac{1}{2}$, $\xi \in \mathbb{R}$, $a = 1$, $b = c = e$, we obtain the integral representation for the Apostol-type Frobenius–Euler polynomials*:

$$\begin{aligned}\mathfrak{H}_n\left(x;1,e,e;u;-ue^{2\pi i\xi}\right) &= \frac{\frac{u-1}{u}e^{-2\xi\pi ix}}{2}\left\{\int_0^\infty \frac{D(n;x,v)(e^{2\pi(v-ix)}e^{2\xi\pi u}+e^{-2\xi\pi v})}{\cosh 2\pi v-\cos 2\pi x}v^n\,dv\right\}\\ &\quad+\frac{\frac{u-1}{u}e^{-2\xi\pi ix}}{2}\left\{\int_0^\infty \frac{iB(n;x,v)e^{2\pi(v-ix)}e^{2\xi\pi v}-e^{-2\xi\pi v})}{\cosh 2\pi v-\cos 2\pi x}v^n\,dv\right\},\end{aligned}$$

where

$$\begin{aligned}D(n;x,v) &= \left[e^{\pi v}\cos\left(\pi x-\frac{(n+1)\pi}{2}\right)-e^{-\pi v}\cos\left(\pi x+\frac{(n+1)\pi}{2}\right)\right],\\ B(n;x,v) &= \left[e^{\pi v}\sin\left(\pi x-\frac{(n+1)\pi}{2}\right)-e^{-\pi v}\sin\left(\pi x+\frac{(n+1)\pi}{2}\right)\right].\end{aligned}$$

Corollary 4.2 *For $n \in \mathbb{N}$ and $0 \le x \le 1$, $|\xi| < \frac{1}{2}$, $\xi \in \mathbb{R}$, $a = 1$, $b = c = e$ $u = -1$, we have*

$$\begin{aligned}\mathfrak{H}_n\left(x;1,e,e;-1;e^{2\pi i\xi}\right) &= e^{-2\xi\pi ix}\left\{\int_0^\infty \frac{D(n;x,v)(e^{2\pi(v-ix)}e^{2\xi\pi v}+e^{-2\xi\pi v})}{\cosh 2\pi v-\cos 2\pi x}v^n\,dv\right\}\\ &\quad+e^{-2\xi\pi ix}\left\{\int_0^\infty \frac{iB(n;x,v)e^{2\pi(v-ix)}e^{2\xi\pi v}-e^{-2\xi\pi v})}{\cosh 2\pi v-\cos 2\pi x}v^n\,dv\right\},\end{aligned}$$

where

$$\begin{aligned}D(n;x,v) &= \left[e^{\pi v}\cos\left(\pi x-\frac{(n+1)\pi}{2}\right)-e^{-\pi v}\cos\left(\pi x+\frac{(n+1)\pi}{2}\right)\right],\\ B(n;x,v) &= \left[e^{\pi v}\sin\left(\pi x-\frac{(n+1)\pi}{2}\right)-e^{-\pi v}\sin\left(\pi x+\frac{(n+1)\pi}{2}\right)\right].\end{aligned}$$

The result obtained in Corollary 4.2 is a new integral representation for the Apostol–Euler polynomials $\mathfrak{H}_n(x;1,e,e;-1;e^{2\pi i\xi}) = \mathfrak{E}_n(x;e^{2\pi i\xi})$.

Next, we obtain the integral representation of Apostol-type Frobenius–Genocchi polynomials.

Theorem 4.2 *For $n \in \mathbb{N}$, $0 \le x \le 1$, $|\xi| < 1/2$, $\xi \in \mathbb{R}$, we have*

$$\begin{aligned}&\mathfrak{G}_n^F\left(x;u;-ue^{2\pi i\xi}\right)\\ &\quad=\left[\frac{u-1}{u}\right]\frac{n}{2}e^{-2\xi\pi ix}\left\{\int_0^\infty \frac{D_1(n;x,v)(e^{2\pi(v-ix)}e^{2\xi\pi v}+e^{-2\xi\pi v})}{\cosh 2\pi v-\cos 2\pi x}v^{n-1}\,dv\right\}\\ &\qquad+\left[\frac{u-1}{u}\right]\frac{n}{2}e^{-2\xi\pi ix}\left\{\int_0^\infty \frac{iB_1(n;x,v)e^{2\pi(v-ix)}e^{2\xi\pi v}-e^{-2\xi\pi v})}{\cosh 2\pi v-\cos 2\pi x}v^{n-1}\,dv\right\},\end{aligned}$$

where

$$D_1(n;x,v) = \left[e^{\pi v}\cos\left(\pi x-\frac{n\pi}{2}\right)-e^{-\pi v}\cos\left(\pi x+\frac{n\pi}{2}\right)\right],$$

$$B_1(n;x,\nu)=\left[e^{\pi\nu}\sin\left(\pi x-\frac{n\pi}{2}\right)-e^{-\pi\nu}\sin\left(\pi x+\frac{n\pi}{2}\right)\right].$$

Proof Considering (8), $\lambda=-ue^{2\pi i\xi}$, $k\longmapsto -k$, then

$$\mathfrak{G}_{\mathrm{n}}{}^{F}\left(x;u;-ue^{2\pi i\xi}\right)=\left[\frac{u-1}{u}\right]e^{-\pi ix}e^{-2\pi i\xi x}\frac{n!}{(-\pi i)^n}\sum_{k\in\mathbb{Z}}\frac{e^{-2\pi ikx}}{[2k+2\xi+1]^n}.$$

Using (5), and noting that $\frac{-1}{i}=e^{n\pi i/2}$, $(-1)^n=e^{-n\pi i}$, then we have

$$\begin{aligned}\mathfrak{G}_{\mathrm{n}}{}^{F}\left(x;u;-ue^{2\pi i\xi}\right)&=\left[\frac{u-1}{u}\right]\frac{n}{(-\pi i)^n}\left\{\sum_{k=0}^{\infty}e^{-(2k+2\xi+1)\pi ix}\int_0^\infty t^{n-1}e^{-(2k+2\xi+1)t}\,dt\right.\\&\quad\left.+(-1)^n\sum_{k=0}^{\infty}e^{-(-2k+2\xi+1)\pi ix}\int_0^\infty t^{n-1}e^{-(2k-2\xi-1)t}\,dt\right\}\\&=\left[\frac{u-1}{u}\right]\frac{n}{(-\pi i)^n}\left\{\int_0^\infty\frac{e^{-(2\xi+1)\pi ix}}{e^{2t}-e^{-2\pi ix}}e^{2t}e^{-(2\xi+1)t}t^{n-1}\,dt\right.\\&\quad\left.+(-1)^n\int_0^\infty\frac{e^{(2\xi+1)\pi ix}}{e^{2t}-e^{2\pi ix}}e^{2t}e^{(1+2\xi)t}t^{n-1}\,dt\right\}\\&=\frac{1}{2}\left[\frac{u-1}{u}\right]\frac{n}{\pi^n}\left\{\int_0^\infty e^{\frac{n\pi i}{2}}\frac{(e^{-2\pi ix}-e^{-2t})}{\cosh 2t-\cos 2\pi x}e^{\pi ix}e^{-(2\xi-1)t}t^{n-1}\,dt\right.\\&\quad\left.+\int_0^\infty e^{\frac{-n\pi i}{2}}\frac{(e^{2\pi ix}-e^{-2t})}{\cosh 2t-\cos 2\pi x}e^{-3\pi ix}e^{(2\xi+3)t}t^n\,dt\right\}.\end{aligned}$$

Using $(\frac{1}{-i})^n=e^{\frac{n\pi i}{2}}$ and $(-1)^n=e^{-n\pi i}$, making the substitution $t=\pi\nu$ and simplifying, we complete the proof. □

Corollary 4.3 *For* $n\in\mathbb{N}$, $0\le x\le 1$, $|\xi|<1/2$, $\xi\in\mathbb{R}$, *and* $u=-1$ *we have*

$$\begin{aligned}\mathfrak{G}_n^F\left(x;-1;e^{2\pi i\xi}\right)&=ne^{-2\xi\pi ix}\left\{\int_0^\infty\frac{D_1(n;x,\nu)(e^{2\pi(\nu-ix)}e^{2\xi\pi\nu}+e^{-2\xi\pi\nu})}{\cosh 2\pi\nu-\cos 2\pi x}\nu^{n-1}\,d\nu\right\}\\&\quad+ne^{-2\xi\pi ix}\left\{\int_0^\infty\frac{iB_1(n;x,\nu)e^{2\pi(\nu-ix)}e^{2\xi\pi\nu}-e^{-2\xi\pi\nu})}{\cosh 2\pi\nu-\cos 2\pi x}\nu^{n-1}\,d\nu\right\},\end{aligned}$$

where

$$\begin{aligned}D_1(n;x,\nu)&=\left[e^{\pi\nu}\cos\left(\pi x-\frac{n\pi}{2}\right)-e^{-\pi\nu}\cos\left(\pi x+\frac{n\pi}{2}\right)\right],\\B_1(n;x,\nu)&=\left[e^{\pi\nu}\sin\left(\pi x-\frac{n\pi}{2}\right)-e^{-\pi\nu}\sin\left(\pi x+\frac{n\pi}{2}\right)\right].\end{aligned}$$

The result obtained in Corollary 4.3, is the integral representation for the Apostol–Genocchi polynomials.

Theorem 4.3 *For* $n\in\mathbb{N}$, $0\le x\le 1$, $|\xi|<1/2$, $\xi\in\mathbb{R}$, *we have*

$$\mathfrak{H}_n\left(x;e^{2\pi i\xi}\right)=\left[1-e^{-2\xi\pi i}\right]\frac{e^{2\xi\pi ix}}{2}\left\{\int_0^\infty\frac{D_2(n;x,\nu)(e^{\pi ix}e^{2\xi\pi\nu}+e^{-\pi ix}e^{-2\xi\pi\nu})}{\cosh 2\pi\nu-\cos 2\pi x}\nu^n\,d\nu\right\}$$

$$+\left[1-e^{-2\xi\pi i}\right]\frac{e^{-2\xi\pi ix}}{2}\left\{\int_0^\infty \frac{iB_2(n;x,v)(e^{\pi ix}e^{2\xi\pi v}-e^{-\pi ix}e^{-2\xi\pi v})}{\cosh 2\pi v-\cos 2\pi x}v^n\,dv\right\},$$

where

$$D_2(n;x,v)=\left[e^{\pi v}\cos\left(\pi x-\frac{(n+1)\pi}{2}\right)-e^{-\pi v}\cos\left(\pi x+\frac{(n+1)\pi}{2}\right)\right],$$
$$B_2(n;x,v)=\left[e^{\pi v}\sin\left(\pi x-\frac{(n+1)\pi}{2}\right)+e^{-\pi v}\sin\left(\pi x+\frac{(n+1)\pi}{2}\right)\right].$$

Proof Returning to (17), setting $u=e^{2\pi i\xi}$, $k\longmapsto -k$ and using the well-known integral formula (5), we complete the proof. □

Theorem 4.4 *For $n\in\mathbb{N}$, $0\le x\le 1$, $|\xi|<1/2$, $\xi\in\mathbb{R}$, we have*

$$\begin{aligned}&\mathfrak{G}_n^F\left(x;e^{2\pi i\xi}\right)\\&=\left[1-e^{-2\xi\pi i}\right]\frac{ne^{2\xi\pi ix}}{2}\left\{\int_0^\infty\frac{D_3(n;x,v)(e^{\pi ix}e^{2\xi\pi v}+e^{-\pi ix}e^{-2\xi\pi v})}{\cosh 2\pi v-\cos 2\pi x}v^{n-1}\,dv\right\}\\&\quad+\left[1-e^{-2\xi\pi i}\right]\frac{ne^{-2\xi\pi ix}}{2}\left\{\int_0^\infty\frac{iB_3(n;x,v)(e^{\pi ix}e^{2\xi\pi v}-e^{-\pi ix}e^{-2\xi\pi v})}{\cosh 2\pi v-\cos 2\pi x}v^{n-1}\,dv\right\},\end{aligned}$$

where

$$D_3(n;x,v)=\left[e^{\pi v}\cos\left(\pi x-\frac{n\pi}{2}\right)-e^{-\pi v}\cos\left(\pi x+\frac{n\pi}{2}\right)\right],$$
$$B_3(n;x,v)=\left[e^{\pi v}\sin\left(\pi x-\frac{n\pi}{2}\right)+e^{-\pi v}\sin\left(\pi x+\frac{n\pi}{2}\right)\right].$$

Proof Returning to (4) and setting $u=e^{2\pi i\xi}$, $k\longmapsto -k$ and using the well-known integral formula (5), we complete the proof. □

5 Explicit formulas for the generalized Apostol–type Frobenius–Euler polynomials at rational arguments

In this section, we show the formula in rational arguments of generalized Apostol-type Frobenius–Euler polynomials, Apostol Frobenius–Euler polynomials, Apostol Frobenius–Genocchi polynomials, Frobenius–Genocchi polynomials, Frobenius–Euler polynomials.

Theorem 5.1 *For $n,q\in\mathbb{N}$, $p\in\mathbb{Z}$, $u\in\mathbb{C}$, with $\Re eu\neq 1$, $\xi\in\mathbb{R}$, $|\xi|<1$, $1\le a\le 1.1$, $b>1$ and $1<c\le e$, we have the formula for the generalized Apostol-type Frobenius–Euler polynomials at rational arguments given by*

$$\begin{aligned}&\mathfrak{H}_n\left(\frac{p}{q};a,b,c;u;-ue^{2\pi i\xi}\right)\\&=A_n(a,b;u)\frac{n!}{(2q\pi)^{n+1}}\left\{\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j+2\xi-1}{2q}\right)e^{(\frac{(n+1)}{2}-\frac{(2j+2\xi+1)p\frac{\ln c}{\ln b}}{q})\pi i}\right.\\&\quad\left.+\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j-2\xi-3}{2q}\right)e^{(\frac{-(n+1)}{2}-\frac{(2j-2\xi-3)p\frac{\ln c}{\ln b}}{q})\pi i}\right\},\end{aligned}$$

where

$$A_n(a,b;u) = (\ln b)^n\left[\frac{u-e^{i\pi\frac{\ln a}{\ln b}}}{u}\right].$$

Proof From Eq. (12) and

$$i^{n+1} = e^{\frac{(n+1)\pi i}{2}}$$

we get

$$\begin{aligned}
&\mathfrak{H}_n(x;a,b,c;u;\lambda)\\
&\quad = n!(\ln b)^n\left(\frac{u}{\lambda}\right)^{x\frac{\ln c}{\ln b}}\left[\frac{u-(\frac{u}{\lambda})^{\frac{\ln a}{\ln b}}}{u}\right]i^{n+1}\left[\sum_{k=0}^{\infty}\frac{e^{((\frac{n+1}{2})\pi-2\pi kx\frac{\ln c}{\ln b})i}}{[2\pi ik+\log(\frac{\lambda}{u})]^{n+1}}\right]\\
&\qquad + n!(\ln b)^n\left(\frac{u}{\lambda}\right)^{x\frac{\ln c}{\ln b}}\left[\frac{u-(\frac{u}{\lambda})^{\frac{\ln a}{\ln b}}}{u}\right]i^{n+1}\left[\sum_{k=0}^{\infty}\frac{e^{(-(\frac{n+1}{2})\pi+2\pi kx\frac{\ln c}{\ln b})i}}{[2\pi ik-\log(\frac{\lambda}{u})]^{n+1}}\right].
\end{aligned} \tag{21}$$

The result shown below is equivalent to (21):

$$\begin{aligned}
&\mathfrak{H}_{\mathrm{n}}(x;a,b,c;u;\lambda)\\
&\quad = n!(\ln b)^n\left(\frac{u}{\lambda}\right)^{x\frac{\ln c}{\ln b}}\left[\frac{u-(\frac{u}{\lambda})^{\frac{\ln a}{\ln b}}}{u}\right]i^{n+1}\left[\sum_{k=1}^{\infty}\frac{e^{((\frac{n+1}{2})\pi-(2k-2)\frac{\ln c}{\ln b}\pi x)i}}{[2\pi ik-2\pi i+\log(\frac{\lambda}{u})]^{n+1}}\right]\\
&\qquad + n!(\ln b)^n\left(\frac{u}{\lambda}\right)^{x\frac{\ln c}{\ln b}}\left[\frac{u-(\frac{u}{\lambda})^{\frac{\ln a}{\ln b}}}{u}\right]i^{n+1}\left[\sum_{k=1}^{\infty}\frac{e^{((-\frac{n+1}{2})\pi+(2k-2)\frac{\ln c}{\ln b}\pi x)i}}{[2\pi ik-2\pi i-\log(\frac{\lambda}{u})]^{n+1}}\right].
\end{aligned} \tag{22}$$

Thus, according to Eq. (10) and by the elementary identity

$$\sum_{k=1}^{\infty}f(k) = \sum_{j=1}^{l}\sum_{k=0}^{\infty}f(lk+j), \quad l\in\mathbb{N}, \tag{23}$$

(see [14, p. 2202, Eq. 4.12]) we find the formula

$$\begin{aligned}
&\mathfrak{H}_{\mathrm{n}}(x;a,b,c;u;\lambda)\\
&\quad = (\ln b)^n\left(\frac{u}{\lambda}\right)^{x\frac{\ln c}{\ln b}}\left[\frac{u-(\frac{u}{\lambda})^{\frac{\ln a}{\ln b}}}{u}\right]\\
&\qquad \times \frac{n!}{(2\pi il)^{n+1}}i^{n+1}\left\{\sum_{j=1}^{l}\Phi\left(e^{-2l\pi x\frac{\ln c}{\ln b}i}, n+1, \frac{2\pi ji+\log(\frac{\lambda}{u})}{2\pi il}\right)\tau\right.\\
&\qquad \left.+\sum_{j=1}^{l}\Phi\left(e^{2l\pi x\frac{\ln c}{\ln b}i}, n+1, \frac{2\pi ji-\log(\frac{\lambda}{u})}{2\pi il}\right)e^{(\frac{-(n+1)\pi}{2}-2\pi x\frac{\ln c}{\ln b}+2j\pi x\frac{\ln c}{\ln b})i}\right\},
\end{aligned} \tag{24}$$

where $\tau = e^{(\frac{(n+1)\pi}{2}+2\pi x\frac{\ln c}{\ln b}-2j\pi x\frac{\ln c}{\ln b})i}$. Setting $\lambda = -ue^{2\pi i\xi}$, $x = \frac{p}{q}$, $l = q$ in (24), the proof of Theorem 5.1 is completed. □

Corollary 5.1 *For $n,q \in \mathbb{N}$, $p \in \mathbb{Z}$, $u \in \mathbb{C}$, with $\Re eu \neq 1$, $\xi \in \mathbb{R}$, $|\xi| < 1$, and $a = 1$, $b = c = e$ we have the following formula at rational arguments for the Apostol Frobenius–Euler polynomials*:

$$\begin{aligned}
&\mathfrak{H}_n\left(\frac{p}{q};1,e,e;u;-ue^{2\pi i\xi}\right)\\
&= \left[\frac{u-1}{u}\right]\frac{n!}{(2q\pi)^{n+1}}\left\{\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j+2\xi-1}{2q}\right)e^{(\frac{(n+1)}{2}-\frac{(2j+2\xi+1)p}{q})\pi i}\right.\\
&\quad\left.+\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j-2\xi-3}{2q}\right)e^{(\frac{-(n+1)}{2}-\frac{(2j-2\xi-3)p}{q})\pi i}\right\}.
\end{aligned}$$

Theorem 5.2 *For $n,q \in \mathbb{N}$, $p \in \mathbb{Z}$, $u \in \mathbb{C}$, with $\Re eu \neq 1$, $\xi \in \mathbb{R}$, $|\xi| < 1$, we have the following formula at rational arguments for the Apostol Frobenius–Genocchi polynomials*:

$$\begin{aligned}
\mathfrak{G}_{\mathrm{n}}{}^{F}\left(\frac{p}{q};u,;-ue^{2\pi i\xi}\right) &= \left[\frac{u-1}{u}\right]\frac{n!}{(2q\pi)^{n}}\left\{\sum_{j=1}^{q}\zeta\left(n,\frac{2j-2\xi-3}{2q}\right)e^{(\frac{(2j-2\xi-2)p}{q}-\frac{n}{2})\pi i}\right.\\
&\quad\left.+\sum_{j=1}^{q}\zeta\left(n,\frac{-2j+2\xi-1}{2q}\right)e^{(-\frac{(2j+2\xi)p}{q}+\frac{n}{2})\pi i}\right\}.
\end{aligned}$$

Theorem 5.3 *For $n,q \in \mathbb{N}$, $p \in \mathbb{Z}$, $\xi \in \mathbb{Z}$, $u \in \mathbb{C}$, $|\xi| < 1$, we have the following formula at rational arguments for the Frobenius–Euler polynomials*:

$$\begin{aligned}
\mathfrak{H}_{\mathrm{n}}\left(\frac{p}{q};e^{2\pi i\xi}\right) &= \left[1-e^{2\pi i\xi}\right]\frac{n!}{(2q\pi)^{n+1}}\left\{\sum_{j=1}^{q}\zeta\left(n+1,\frac{j+\xi-1}{q}\right)e^{(\frac{(2j+2\xi-1)p}{q}-\frac{(n+1)}{2})\pi i}\right.\\
&\quad\left.+\sum_{j=1}^{q}\zeta\left(n+1,\frac{j-\xi-1}{2q}\right)e^{(-\frac{(2j-2\xi-1)p}{q}+\frac{(n+1)}{2})\pi i}\right\}.
\end{aligned}$$

Theorem 5.4 *For $n,q \in \mathbb{N}$, $p \in \mathbb{Z}$, $u \in \mathbb{C}$, with $\Re eu \neq 1$, $\xi \in \mathbb{R}$, $|\xi| < 1$,, the following formula at rational arguments of Frobenius–Genocchi polynomials*:

$$\begin{aligned}
\mathfrak{G}_{\mathrm{n}}{}^{F}\left(\frac{p}{q};e^{2\pi i\xi}\right) &= \left[1-e^{2\pi i\xi}\right]\frac{n!}{(2q\pi)^{n}}\left\{\sum_{j=1}^{q}\zeta\left(n,\frac{j+\xi-1}{q}\right)e^{(\frac{(2j+2\xi-1)p}{q}-\frac{n}{2})\pi i}\right.\\
&\quad\left.+\sum_{j=1}^{q}\zeta\left(n,\frac{j-\xi-1}{2q}\right)e^{(-\frac{(2j-2\xi-1)p}{q}+\frac{n}{2})\pi i}\right\}.
\end{aligned}$$

For the proof of Theorems 5.2, 5.3 *and* 5.4 *we use* (8), (2) *and* (10), (4) *and the identity* (23).

6 Conclusions

In this article, we showed the Fourier series representation of generalized Apostol-type Frobenius–Euler polynomials by using the proof of the Cauchy residue theorem. The result presented generalizes several Fourier series representations for polynomial families known to date. Also, we proved an integral representation for this and other known polynomial families. Finally, we presented the explicit formula in rational arguments in terms of the Zeta Hurwit Lerch and Zeta Hurwit functions for the generalized Apostol-type Frobenius Euler polynomials also said to be of Euler type.

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Author details
[1]Programa de Matemáticas, Universidad del Atlántico, Km 7 Vía Pto, Barranquilla, Colombia. [2]Departamento de Ciencias Naturales y Exactas, Universidad de la Costa, Barranquilla, Colombia. [3]Departamento de Ciencias Básicas, Universidad Metropolitana, Barranquilla, Colombia.

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